RIESZ BASES OF REPRODUCING KERNELS IN FOCK TYPE SPACES

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ABSTRACT. In a scale of Fock spaces $\mathcal{F}_\varphi$ with radial weights φ we study the existence of Riesz bases of (normalized) reproducing kernels. We prove that these spaces possess such bases if and only if $\varphi(x)$ grows at most like $(\log x)^2$.

1. INTRODUCTION.

Given an increasing function φ defined on $[0,+\infty)$, we extend it to $\mathbb{C}$ by $\varphi(z)=\varphi(|z|)$, and consider the Fock type space

$$\mathcal{F}_\varphi=\{f\in \mathrm{Hol}\,(\mathbb{C}\,):\|f\|^2_\varphi=\int_{\mathbb{C}}|f(z)|^2e^{-2\varphi(z)}dm(z)<\infty\},$$

where dm is the area Lebesgue measure.

The Hilbert space $\mathcal{F}_\varphi$ possesses the bounded point evaluation property, i.e., for each $\lambda\in\mathbb{C}$, the mapping $L_\lambda: f\mapsto f(\lambda)$ is a bounded linear functional in $\mathcal{F}_\varphi$. Therefore there exists $\mathbf{k}_\lambda=\mathbf{k}^\varphi_\lambda\in\mathcal{F}_\varphi$, the reproducing kernel at λ in $\mathcal{F}_\varphi$:

$$f(\lambda)=\langle f,\mathbf{k}_\lambda\rangle_\varphi,\qquad \lambda\in\mathbb{C}\,,\ f\in\mathcal{F}_\varphi,$$

and we have

$$\|L_\lambda\|_{\mathcal{F}_\varphi\to\mathbb{C}}=\|\mathbf{k}_\lambda\|_\varphi=\big(\mathbf{k}_\lambda(\lambda)\big)^{1/2}.$$

Let $\mathbb{k}_\lambda=\mathbf{k}_\lambda/\|\mathbf{k}_\lambda\|_\varphi$ be the normalized reproducing kernel at λ. Given a sequence $\Lambda\subset\mathbb{C}$, we say that $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_\varphi$ if it is complete and for some $c,C>0$ we have

$$c\sum_{\lambda\in\Lambda}|a_\lambda|^2\le\Big\|\sum_{\lambda\in\Lambda}a_\lambda\mathbb{k}_\lambda\Big\|^2_\varphi\le C\sum_{\lambda\in\Lambda}|a_\lambda|^2,$$

for each finite sequence $\{a_\lambda\}\subset\mathbb{C}$. Equivalently $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a linear isomorphic image of an orthonormal basis in a separable Hilbert space.

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In this article we study the following question: *for which φ does the space $\mathcal{F}_\varphi$ admit a Riesz basis of normalized reproducing kernels?*

The above question can be reformulated in the classical terms of interpolation in Hilbert spaces of entire functions. Let X be such a space and let $\mathbf{k}_\lambda^X$ stand for the reproducing kernel at λ in X. We say that a sequence $\Lambda \subset \mathbb{C}$ is a *complete interpolating sequence for* X if, for each $\{a_\lambda\} \in \ell^2(\Lambda)$, the interpolation problem

$$\frac{f(\lambda)}{\|\mathbf{k}_\lambda^X\|_X} = a_\lambda, \qquad \lambda \in \Lambda, \tag{1.1}$$

has a unique solution in X.

Standard duality arguments show that the system $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in X if and only if Λ is a complete interpolating sequence for this space, $\mathbb{k}_\lambda = \mathbf{k}_\lambda^X/\|\mathbf{k}_\lambda^X\|_X$.

A canonical example here is the Paley-Wiener space, i.e. the space of Fourier transforms of all functions from $L^2(-\pi,\pi)$. The set $\Lambda = \mathbb{Z}$ is a complete interpolating sequence for this space. Furthermore, a notoriously difficult open problem by Nikolski is whether every model space $K_\Theta = H^2 \ominus \Theta H^2$ (Θ is an inner function) has a Riesz basis of reproducing kernels. We refer also to the papers [4, 3, 6] studying expansions in exponential series on different domains of the complex plane. These problems can be reformulated in terms of bases of reproducing kernels in Fock type spaces. In particular, the results of [3] show that $\mathcal{F}_\varphi$ with $\varphi(x) = x - \frac{3}{2}\log x$ has no Riesz basis of (normalized) reproducing kernels.

Starting from the results of Seip [10], it is known that the classical Fock space, $\mathcal{F}_\varphi$ with $\varphi(x) = x^2$, unlike the Paley–Wiener spaces, has no Riesz basis of (normalized) reproducing kernels.

For more rapidly growing φ the absence of such bases is established in [1] (see Theorems 2.2 and 2.4 therein) under some natural regularity conditions on ϕ. On the other hand, if $\varphi(x) = \mathrm{Const}\cdot\log|x|$, then $\mathcal{F}_\varphi$ becomes a finite dimensional space of polynomials so that each $\Lambda \subset \mathbb{C}$ with $\mathrm{Card}(\Lambda) = \dim(\mathcal{F}_\varphi)$ is obviously a complete interpolating sequence for this space.

In this paper we assume that $\varphi(z) = \varphi(|z|)$ is a subharmonic function such that

$$\log|x| = o(\varphi(x)), \qquad x \to \infty,$$

hence $\dim \mathcal{F}_\varphi = \infty$, and study what happens if $\varphi(x)$ grows less rapidly than x^2.

It turns out that for $\varphi(x) = (\log x)^\beta$, $1 < \beta \le 2$, the spaces $\mathcal{F}_\varphi$ still have Riesz bases of normalized reproducing kernels. On the other hand, for φ such that $(\log x)^2 = o(\varphi(x))$, $x \to \infty$, the spaces $\mathcal{F}_\varphi$ have

no such bases (again assuming that ϕ satisfies some natural regularity conditions). Roughly speaking, the reason for this is that the local scale function $\rho(z) = (\Delta\varphi(z))^{-1/2}$ is $o(|z|)$, $z \to \infty$, if $(\log x)^2 = o(\varphi(x))$, $x \to \infty$, and is comparable to z, if $\varphi(x) = (\log x)^2$.

Precise formulations of our results are given in Section 2. We study the case when both $(\log x)^2 = o(\varphi(x))$ and $\varphi(x) = O(x^2)$ in Section 3. There we apply a theorem from [5] in order to approximate $\exp(\varphi)$ by the modulus of an entire function using discretization of the Riesz measure $d\mu_\varphi = \triangle\varphi(z)\,dm(z)$ of the subharmonic function φ, and then use an argument by Seip from [10, Lemma 6.2]. In Section 4 we deal with the borderline case $\varphi(x) = (\log x)^2$, and again use approximation of $\exp(\varphi)$ by the modulus of an entire function. Finally, our argument in Section 5 dealing with the case $\varphi(x) = x^\beta$, $1 < \beta < 2$, is essentially a real variable one using Legendre transform estimates.

In Section 2 we obtain asymptotic estimates (Lemma 2.3, Lemma 2.7) of the norm of the reproducing kernel coinciding with those by Holland–Rochberg in [2]. We cannot just refer to [2] since the conditions imposed on φ there are not satisfied in our situation.

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2. Formulation of the results

Part A. In this subsection we deal with regular functions $\varphi(x)$ growing more rapidly than $(\log x)^2$, but less rapidly than x^2. We assume that $\varphi(z) = \varphi(|z|)$ is C^2-smooth and subharmonic on $\mathbb{C}$, and set

$$\rho(z) = \big(\triangle\varphi(z)\big)^{-1/2} = \Big(\frac{\varphi'(r)}{r} + \varphi''(r)\Big)^{-1/2},$$

where $r = |z|$.

The function $\rho(z)$ defines a natural scale with respect to the Riesz measure μ_φ of φ, i.e. $\mu_\varphi(\{\zeta : |\zeta - z| \le \rho(z)\}) \asymp 1$ for all $z \in \mathbb{C}$. Here and in what follows, the notation $A(s) \prec B(s)$ for s in some set S means that the ratio $A(s)/B(s)$ of the two positive functions $A(s)$ and $B(s)$ is bounded from above by a positive constant independent of s in S. We write $A(s) \succ B(s)$ if $B(s) \prec A(s)$ and $A(s) \asymp B(s)$ if both $A(s) \prec B(s)$ and $B(s) \prec A(s)$.

In the borderline cases $\varphi(x) = x^2$ and $\varphi(x) = (\log x)^2$ we have respectively $\rho(r) = \text{Const}$ and $\rho(r) = \text{Const} \cdot r$.

In this subsection we assume that

$$0<\inf_{r>0}\rho(r),\ \text{and}\ \rho(r)=o(r),\quad r\to\infty, \tag{2.1}$$

and also

$$\left.\begin{aligned}\rho(r+\rho(r))=(1+o(1))\rho(r),\quad r\to\infty,\\ \rho(2r)\asymp\rho(r),\qquad r>0.\end{aligned}\right\} \tag{2.2}$$

Lemma 2.1. *Given $w\in\mathbb{C}$, there exists a function Φ_w analytic in the disc $D_w=\{z\in\mathbb{C}:|z-w|<\rho(w)\}$ and such that*

$$|\Phi_w(z)|\asymp e^{\varphi(z)},\qquad z\in D_w.$$

Lemma 2.2. *There exists an entire function F such that*

$$|F(z)|\asymp e^{\varphi(z)}\cdot\frac{\operatorname{dist}(z,W)}{\rho(z)},\qquad z\in\mathbb{C}\,, \tag{2.3}$$

where W is the zero set of F, and

$$\operatorname{dist}(w,W\setminus\{w\})\succ\rho(w),\quad w\in W,$$
$$\operatorname{dist}(z,W)\prec\rho(z),\quad z\in\mathbb{C}\,.$$

Lemma 2.3.

$$\|\mathbf{k}_z\|^2_\varphi\asymp e^{2\varphi(z)}/\rho^2(z),\qquad z\in\mathbb{C}\,.$$

Lemma 2.4. *Let a sequence $\Lambda\subset\mathbb{C}$ be such that $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_\varphi$. Then*

(a) $\operatorname{dist}(\lambda,\Lambda\setminus\{\lambda\})\succ\rho(\lambda)$, $\lambda\in\Lambda$,
(b) $\operatorname{dist}(z,\Lambda)\prec\rho(z)$, $z\in\mathbb{C}\,$.

Theorem 2.5. *Under conditions* (2.1), (2.2) *the space $\mathcal{F}_\varphi$ has no Riesz bases of normalized reproducing kernels.*

Part B. In this subsection we assume that $\varphi(r)=(\log^+r)^2$.

Lemma 2.6. *Let $\Lambda=\{\exp(\frac{n+1}{2}+i\theta_n)\}_{n\ge0}$, where θ_n are arbitrary real numbers. The product*

$$E(z)=\prod_{\lambda\in\Lambda}\Big(1-\frac{z}{\lambda}\Big)$$

converges uniformly on compact sets in $\mathbb{C}$ and satisfies the estimate

$$|E(z)|\asymp e^{\varphi(z)}\cdot\frac{\operatorname{dist}(z,\Lambda)}{|z|^{3/2}},\qquad z\in\mathbb{C}\,. \tag{2.4}$$

Lemma 2.7.

$$\|\mathbf{k}_z\|_\varphi^2 \asymp e^{2\varphi(z)}/(1+|z|^2), \qquad z \in \mathbb{C}.$$

Theorem 2.8. *Let $\varphi(r) = (\log^+ r)^2$, and let Λ be as in Lemma 2.6. Then $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_\varphi$.*

Part C. In this subsection we consider the case $\varphi(r) = (\log^+ r)^{1+\delta}$ for $0 < \delta < 1$.

Denote $w_n = \log \|z^n\|_\varphi^2$, $n \ge 0$. Then

$$\mathbf{k}_\lambda(z) = \sum_{n\ge 0} \bar{\lambda}^n z^n e^{-w_n}$$

and

$$\|\mathbf{k}_\lambda\|_\varphi^2 = \sum_{n\ge 0} |\lambda|^{2n} e^{-w_n}.$$

Lemma 2.9. *For some $c > 0$ we have*

$$w_n = c(n+1)^{1+1/\delta} + O\big(\log n\big), \qquad n > 0.$$

Denote $r_0 = 0$, $r_n = \exp[(w_{n+1} - w_{n-1})/4]$, $n \ge 1$.

Theorem 2.10. *Let $0 < \delta < 1$, $\varphi(r) = (\log^+ r)^{1+\delta}$, and let $\lambda_n = r_n e^{i\theta_n}$ with arbitrary real θ_n. Then $\{\mathbb{k}_{\lambda_n}\}_{n\ge 0}$ is a Riesz basis in $\mathcal{F}_\varphi$.*

3. Proofs. Part A.

Proof of Lemma 2.1. (See also [1, Lemma 4.1].) By (2.2) we know that $\rho(z) \asymp \rho(w)$, $z \in D_w$. Set $H(\zeta) = \varphi(w + \zeta\rho(w))$, $|\zeta| \le 1$. Then $\Delta H(\zeta) \asymp 1$, $|\zeta| \le 1$. Next we define

$$G(z) = \int_{\mathbb{D}} \log\left|\frac{\zeta - z}{1 - \bar{\zeta} z}\right| \Delta H(\zeta)\, dm(z), \qquad |z| \le 1.$$

Then $|G(z)| \prec 1$, $|z| \le 1$ uniformly with respect $w \in \mathbb{C}$, and $H_1 = H - G$ is real and harmonic. Denote by $\widetilde{H}_1$ the harmonic conjugate to H_1, $\widetilde{H}_1(0) = 0$, set $H_0 = H_1 + i\widetilde{H}_1$, and define

$$\Phi_w(z) = \exp H_0\big((z - w)/\rho(w)\big).$$

Since $\log|\Phi_w(z)| - \varphi(z) = -G\big((z - w)/\rho(w)\big)$, $z \in D_w$, the proof is completed. $\square$

Proof of Lemma 2.2. This lemma is a special case of Theorem 3 in [5]. We describe just the idea of its proof. It relies on the atomization procedure for the measure μ_φ. In this procedure the complex plane is decomposed into a disjoint union of pieces ω of μ_φ-measure two each, and then an atomized measure $\mu_\varphi^{(a)}$ is constructed; this measure is the sum of discrete unit masses, two masses are situated in each piece ω in such a way that the first two moments of their sum coincide with the corresponding moments of $\mu_\varphi|\omega$. We refer the reader to [11, 9] for other implementation of atomization techniques. $\square$

Proof of Lemma 2.3. We use the fact that $\|\mathbf{k}_z\|_\varphi = \|L_z\|_{\mathcal{F}_\varphi\to\mathbb{C}}$. Given $z \in \mathbb{C}$, take $w, w' \in W$, $w \neq w'$, such that $|z-w| = \operatorname{dist}(\lambda, W)$, $|z-w'| \asymp \rho(z)$, and consider the function $G = F/[(\cdot - w)(\cdot - w')]$, where F is the function from Lemma 2.2, and W is its zero set.

Then

$$|G(z)| \asymp e^{\varphi(z)}/\rho^2(z),$$

$$\|G\|_\varphi^2 \prec \int_{\mathbb{C}} \frac{dm(\zeta)}{\rho^4(z)+|\zeta-z|^4} \asymp \frac{1}{\rho^2(z)}. \tag{3.1}$$

Therefore,

$$\|\mathbf{k}_z^\varphi\|_\varphi^2 \succ e^{2\varphi(z)}/\rho^2(z).$$

Now take any $f \in \mathcal{F}_\varphi$, $\|f\|_\varphi = 1$, and define Φ_z, D_z as in Lemma 2.1. We have

$$\int_{D_z} \big|f(\zeta)/\Phi_z(\zeta)\big|^2 dm(\zeta) \prec \|f\|_\varphi^2 = 1,$$

and the mean value theorem yields

$$|f(z)| \prec |\Phi_z(z)|/\rho(z) \asymp e^{\varphi(z)}/\rho(z).$$

Thus,

$$\|\mathbf{k}_z\|_\varphi \asymp e^{\varphi(z)}/\rho(z).$$

$\square$

Now let, for a sequence $\Lambda \subset \mathbb{C}$, the system $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ be a Riesz basis in $\mathcal{F}_\varphi$. Then the mapping

$$f \mapsto \{\langle f, \mathbf{k}_\lambda\rangle_\varphi\}_{\lambda\in\Lambda} = \{f(\lambda)\}_{\lambda\in\Lambda}$$

is an isomorphism between $\mathcal{F}_\varphi$ and the space

$$\ell^2(1/\|\mathbf{k}_\lambda\|) = \Big\{\{c_\lambda\}_{\lambda\in\Lambda} :\ \|\{c_\lambda\}\|_{\ell^2(1/\|\mathbf{k}_\lambda\|)} = \sum_{\lambda\in\Lambda} |c_\lambda|^2/\|\mathbf{k}_\lambda\|_\varphi^2 < \infty\Big\}.$$

In particular, the interpolating problem

$$f(\lambda) = c_\lambda, \qquad \lambda \in \Lambda, \ f \in \mathcal{F}_\varphi \tag{3.2}$$

has a unique solution for each $\{c_\lambda\} \in \ell^2(1/\|\mathbf{k}_\lambda\|)$ and

$$\|f\|_\varphi \asymp \|f|\Lambda\|_{\ell^2(1/\|\mathbf{k}_\lambda\|)}, \qquad f \in \mathcal{F}_\varphi. \tag{3.3}$$

Fix $\lambda \in \Lambda$ and consider the function $f_\lambda \in \mathcal{F}_\varphi$, solving the interpolating problem $f_\lambda(\mu) = \delta_{\lambda,\mu}$, $\mu \in \Lambda$, here $\delta_{\lambda,\mu}$ is the Kronecker delta function. Then the function $E(z) = (z-\lambda)f_\lambda(z)$ vanishes precisely on Λ (otherwise, Λ would not be a uniqueness set), and the solution to the problem (3.2) has the form

$$f(z) = \sum_{\lambda\in\Lambda} c_\lambda \frac{E(z)}{E'(\lambda)(z-\lambda)},$$

the sum being convergent in $\mathcal{F}_\varphi$.

Proof of Lemma 2.4. (a) Suppose that $0 < |\lambda - \lambda'| \le \rho(\lambda')/N$, $\lambda, \lambda' \in \Lambda$. Denote $D = D_{\lambda'}$, and for $\zeta \in \mathbb{C}$ set

$$f(z) = \frac{E(z)}{z-\lambda}, \qquad f_1(z) = \frac{E(z)}{(z-\lambda)(z-\lambda')}(z - \lambda' - \zeta\rho(\lambda')/2).$$

Then for some $\zeta \in \mathbb{C}$, $|\zeta| = 1$, we have

$$\|f_1\|_\varphi \asymp \|f\|_\varphi. \tag{3.4}$$

Indeed, the corresponding integrals outside D are equivalent. Furthermore, let

$$g(z) = \frac{f(z)}{z-\lambda'} \cdot \frac{1}{\Phi_{\lambda'}(z)}.$$

Then

$$\int_D |f(z)|^2 e^{-2\varphi(z)} dm(z) \asymp \int_D |g(z)|^2 |z-\lambda'|^2 dm(z),$$

and, for an appropriate ζ, $|\zeta| = 1$,

$$\int_D |f_1(z)|^2 e^{-2\varphi(z)} dm(z) \asymp \int_D |g(z)|^2 |z - \lambda' - \frac{1}{2}\zeta\rho(\lambda')|^2 dm(z) =$$
$$\int_D |g(z)|^2 |z-\lambda'|^2 dm(z) + \frac{\rho(\lambda')^2}{4} \int_D |g(z)|^2 dm(z).$$

It remains to use the relation

$$\int_D |g(z)|^2 dm(z) \asymp \rho(\lambda')^{-2} \int_D |g(z)|^2 |z-\lambda'|^2 dm(z).$$

to get (3.4).

However, $\|f_1|\Lambda\|_{\ell^2(1/\|\mathbf{k}_\lambda\|)} \ge CN\|f|\Lambda\|_{\ell^2(1/\|\mathbf{k}_\lambda\|)}$, and we get a contradiction to (3.4) for large N.

(b) Suppose that $\operatorname{dist}(z,\Lambda) \ge N\rho(z)$. Consider the function G from the proof of Lemma 2.3. Using Lemmas 2.1, 2.3, and 2.4 (a) we obtain that for large N,

$$\begin{aligned}\|G|\Lambda\|^2_{\ell^2(1/\|\mathbf{k}_\lambda\|)} &= \sum_{\lambda\in\Lambda}\frac{|G(\lambda)|^2}{\|\mathbf{k}_\lambda\|^2_\varphi} \prec \sum_{\lambda\in\Lambda}|G(\lambda)|^2e^{-2\varphi(\lambda)}(\rho(\lambda))^2\\ &\prec \sum_{\lambda\in\Lambda}\int_{|\zeta-\lambda|<\rho(\lambda)}|G(\zeta)|^2e^{-2\varphi(\zeta)}dm(\zeta)\\ &\prec \int_{|\zeta-z|>N\rho(z)/2}|G(w)|^2e^{-2\varphi(w)}dm(w) \prec \frac{1}{N^2(\rho(z))^2}.\end{aligned}$$

This contradicts to (3.1) for large N. $\square$

Proof of Theorem 2.5. Suppose that the system $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis for $\mathcal{F}_\varphi$. Relation (3.3) and Lemma 2.3 imply that

$$\Big\|\frac{E}{\cdot-\lambda}\Big\|^2_\varphi \asymp |E'(\lambda)|^2\rho^2(\lambda)e^{-2\varphi(\lambda)}, \qquad \lambda\in\Lambda.$$

Consider the function $E/[(\cdot-\lambda)\Phi_\lambda]$. Applying the mean value property we obtain

$$\int_{\mathbb{C}}\frac{|E(z)|^2}{|z-\lambda|^2}e^{-2\varphi(z)}dm(z) \prec \rho^2(\lambda)\frac{1}{\rho^4(\lambda)}\int_{|z-\lambda|<\rho(\lambda)}|E(z)|^2e^{-2\varphi(z)}dm(z).$$

Take large N and $w\in\mathbb{C}$ such that

$$N \ll |w|/\rho(w). \tag{3.5}$$

By Lemma 2.4 (a), we have

$$\begin{aligned}&\int_{\mathbb{C}}\Big(\sum_{|\lambda-w|<N\rho(w),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\Big)|E(z)|^2e^{-2\varphi(z)}dm(z)\\ &\qquad\prec \int_{|w-z|<(N+2)\rho(w)}\rho^{-2}(z)|E(z)|^2e^{-2\varphi(z)}dm(z),\end{aligned}$$

and, as a result,

$$\inf_{z:|z-w|<(N+2)\rho(w)}\Big[\rho^2(z)\Big(\sum_{|\lambda-w|<N\rho(w),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\Big)\Big] \prec 1. \tag{3.6}$$

Finally, by Lemma 2.4 (b),

$$\rho^2(z)\Big(\sum_{|\lambda-w|<N\rho(w),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\Big) \succ \int_{\rho(w)<|\zeta|<N\rho(w)}\frac{dm(\zeta)}{|z-\zeta|^2}.$$

For large N we get a contradiction to (3.6). $\square$

Remark *The above construction became possible due to the fact that one can choose a sufficiently large N satisfying* (3.5). *This is not the case in the situations considered in Parts* B *and* C.

4. Proofs. Part B.

Proof of Lemma 2.6. If $\Lambda = \{\lambda_n\}_{n\ge 0}$, $|\lambda_n| = \exp[(n+1)/2]$, $|z| = \exp t$, and

$$\frac{m}{2} - \frac{1}{4} \le t < \frac{m}{2} + \frac{1}{4},$$

then

$$\begin{aligned}
\log|E(z)| &= \sum_{0\le k<m-1} \log\frac{|z|}{|\lambda_k|} + \log\Big|1 - \frac{z}{\lambda_{m-1}}\Big| + O(1) \\
&= \sum_{0\le k<m} (t-(k+1)/2) + \log \operatorname{dist}(z,\Lambda) - t + O(1) \\
&= mt - m(m+1)/4 + \log \operatorname{dist}(z,\Lambda) - t + O(1) \\
&= t^2 - 3t/2 + \log \operatorname{dist}(z,\Lambda) + O(1), \qquad |z|\to\infty.
\end{aligned}$$

$\square$

We will also need the estimate

$$|E'(\lambda_n)| \asymp |\lambda_n|^{-3/2} e^{\varphi(\lambda_n)}, \tag{4.1}$$

which can be obtained similarly.

Proof of Lemma 2.7. Given $\lambda\in\mathbb{C}$ such that $r = \log|\lambda| \ge 1$, choose n such that $n \le 2r < n+1$, and set $f(z) = z^n$. Then

$$\begin{aligned}
\|f\|_\varphi^2 &= \int_{\mathbb{C}} |z|^{2n} e^{-2\varphi(z)} dm(z) \asymp \int_0^\infty e^{2(n+1)t-2t^2} dt \\
&= e^{(n+1)^2/2} \int_{-(n+1)/2}^\infty e^{-2s^2} ds \prec e^{2(n+1)r-2r^2}.
\end{aligned}$$

Furthermore, $|f(\lambda)| = e^{nr}$, and we conclude that

$$\|\mathbf{k}_\lambda\|_\varphi^2 \succ e^{2r^2-2r}.$$

The opposite inequality is proved as in Lemma 2.3. $\square$

Proof of Theorem 2.8. It suffices to prove that the mapping $f \mapsto f|\Lambda$ is an isomorphism between $\mathcal{F}_\varphi$ and $\ell^2(1/\|\mathbf{k}_\lambda\|)$.

It is straightforward that this mapping is bounded. Indeed, if $f \in \mathcal{F}_\varphi$, then the mean value theorem and Lemma 2.1 yield

$$|f(\lambda_k)|^2 e^{-2\varphi(\lambda_k)}|\lambda_k|^2 \prec \int_{|z-\lambda_k|<|\lambda_k|/10} |f(z)|^2 e^{-2\varphi(z)} dm(z),$$

Since the discs $|z-\lambda_k| < |\lambda_k|/10$ are disjoint we obtain

$$\begin{aligned} &\|f|\Lambda\|^2_{\ell^2(1/\|\mathbf{k}_\lambda\|)} \\ &\qquad \prec \sum_k \int_{|z-\lambda_k|<|\lambda_k|/10} |f(z)|^2 e^{-2\varphi(z)} dm(z) \le \|f\|^2_\varphi, \qquad f \in \mathcal{F}_\varphi. \end{aligned}$$

It is also straightforward that the mapping $f \mapsto f|\Lambda$ has zero kernel. Were this not the case we could take a non-zero $f \in \mathcal{F}_\varphi$ which vanishes on Λ and note that by Lemmas 2.6 and 2.7 the entire function $g = f/E$ satisfies $|g(z)| \prec 1 + |z|^{1/2}$ if $\operatorname{dist}(z,\Lambda) > |z|/10$. The latter restriction can be removed just by the maximum principle, so, by the Liouville theorem we have $g(z) = C$ or $f(z) = CE(z)$ for some constant C. Now we see that $C = 0$, otherwise $f \notin \mathcal{F}_\varphi$, thus arriving to a contradiction.

It remains to prove that the mapping $f \mapsto f|\Lambda$ acts onto $\ell^2(1/\|\mathbf{k}_\lambda\|)$, i.e. the interpolation problem (1.1) has a (unique) solution for each $\{a_\lambda\} \in \ell^2(\Lambda)$.

For finite sequences $\{a_\lambda\}$ the solution is given by the mapping

$$T_\Lambda : \{a_\lambda\} \ \mapsto \ T_\Lambda\{a_\lambda\}(z) = \sum_{\lambda\in\Lambda} a_\lambda \|\mathbf{k}_\lambda\|_\varphi \frac{E(z)}{E'(\lambda)(z-\lambda)}.$$

We will prove that

$$\|T_\Lambda\{a_\lambda\}\|_\varphi \prec \|\{a_\lambda\}\|_{\ell^2(\Lambda)}, \tag{4.2}$$

and then T_Λ extends continuously to the whole $\ell^2(\Lambda)$.

Denote

$$E_\lambda(z) = \|\mathbf{k}_\lambda\|_\varphi \frac{E(z)}{E'(\lambda)(z-\lambda)}, \qquad \lambda \in \Lambda.$$

Relation (4.2) obviously follows from the inequalities

$$\|E_\lambda\|_\varphi \prec 1, \qquad \lambda \in \Lambda, \tag{4.3}$$

and, for some $c > 0$,

$$\big|\langle E_{\lambda_m}, E_{\lambda_n}\rangle_\varphi\big| \prec e^{-c|n-m|}, \qquad m,n \ge 0. \tag{4.4}$$

We are now proving these inequalities. It follows from Lemma 2.7 and (4.1) that

$$\frac{\|\mathbf{k}_{\lambda_n}\|_\varphi}{|E'(\lambda)|} \asymp |\lambda|^{1/2}, \qquad \lambda \in \Lambda.$$

Together with (2.4) this yields

$$|E_\lambda(z)| \asymp |\lambda|^{1/2} \frac{\operatorname{dist}(z,\Lambda)}{(1+|z|^{3/2})|z-\lambda|} e^{\varphi(z)},$$

and

$$\|E_\lambda\|_\varphi^2 \asymp \int_{\mathbb{C}} \frac{|\lambda| \operatorname{dist}(z,\Lambda)^2}{(1+|z|^3)|z-\lambda|^2} dm(z)$$

$$= \left\{ \int_{|z|\le|\lambda|/2} + \int_{|\lambda|/2<|z|<2|\lambda|} + \int_{|z|>2|\lambda|} \right\} \frac{|\lambda| \operatorname{dist}(z,\Lambda)^2}{(1+|z|^3)|z-\lambda|^2} dm(z)$$

$$= I_1 + I_2 + I_3.$$

For $|z| < |\lambda|/2$ we use that $\operatorname{dist}(z,\Lambda) \prec |z|$, $|z-\lambda| \asymp |\lambda|$ to get

$$I_1 \prec \frac{1}{|\lambda|} \int_{|z|<|\lambda|/2} \frac{1}{1+|z|} dm(z) \asymp 1.$$

For $|\lambda|/2 < |z| < 2|\lambda|$ we use that $\operatorname{dist}(z,\Lambda) \prec |z-\lambda|$, $|z| \asymp |\lambda|$ to get

$$I_2 \asymp \frac{1}{|\lambda|^2} \int_{|\lambda|/2<|z|<2|\lambda|} dm(z) \asymp 1.$$

Finally, when $|z| > 2|\lambda|$ we use that $\operatorname{dist}(z,\Lambda) \prec |z|$, $|z-\lambda| \asymp |z|$ to get

$$I_3 \prec |\lambda| \int_{|z|>2|\lambda|} \frac{dm(z)}{1+|z|^3} \asymp 1.$$

A combination of these estimates yields (4.3).

Furthermore, if $0 \le k < m$, $\gamma = e^{1/4}$, then

$$|\langle E_{\lambda_k}, E_{\lambda_m}\rangle_{\mathcal{F}_\varphi}| \prec \int_{\mathbb{C}} \frac{\operatorname{dist}(z,\Lambda)^2}{1+|z|^3} \cdot \frac{|\lambda_k|^{1/2}|\lambda_m|^{1/2}}{|\lambda_k - z||\lambda_m - z|} dm(z)$$

$$= \int_{|z|<|\lambda_k|/\gamma} \dots + \int_{|\lambda_k|/\gamma<|z|<\gamma|\lambda_k|} \dots + \int_{\gamma|\lambda_k|<|z|<|\lambda_m|/\gamma} \dots$$

$$+ \int_{|\lambda_m|/\gamma<|z|<\gamma|\lambda_m|} \dots + \int_{|z|>\gamma|\lambda_m|} \dots \asymp \frac{|\lambda_k|^{1/2}}{|\lambda_m|^{1/2}} \left(1 + \log|\lambda_m/\lambda_k|^{1/2}\right)$$

$$\le c e^{-|k-m|/5},$$

which gives (4.4). $\square$

5. Proofs. Part C.

Proof of Lemma 2.9. We have

$$e^{w_n}=\int_0^\infty r^{2n}e^{-2(\log^+ r)^{1+\delta}}2\pi r\,dr\asymp\int_0^\infty e^{(2n+2)s-2s^{1+\delta}}ds.$$

Thus, to prove the lemma, we need to describe the asymptotic behavior of the integral

$$\int_0^\infty e^{as-s^{1+\delta}}ds$$

as $a\to+\infty$.

This can be done by applying standard asymptotic techniques, so we omit calculations. $\square$

Proof of Theorem 2.10. We need the following auxiliary statement

Lemma 5.1. *The numbers* $r_n=\exp[(w_{n+1}-w_{n-1})/4]$ *satisfy the inequality*

$$r_n^{2n}e^{-w_n}\succ(n+1)^2(s+1)^2r_n^{2s}e^{-w_s},\qquad n>0,\ s\neq n. \tag{5.1}$$

Proof. It suffices to prove that, for each $A>0$, there exists a constant C such that

$$\begin{gathered}\frac{n-s}{2}\left[(n+2)^{1+1/\delta}-n^{1+1/\delta}\right]+(s+1)^{1+1/\delta}-(n+1)^{1+1/\delta}\geq\\ C+A|n-s|\log(n+2)+A\log(s+2),\quad n>0,\ s\geq 0,\ n\neq s.\end{gathered}\tag{5.2}$$

Then inequality (5.1) will follow from Lemma 2.9.

Let $n>s$. Denote $\omega(t)=t^{1+1/\delta}$. We have

$$\begin{aligned}&\frac{n-s}{2}\left[(n+2)^{1+1/\delta}-n^{1+1/\delta}\right]+(s+1)^{1+1/\delta}-(n+1)^{1+1/\delta}=\\ &\qquad\frac{n-s}{2}\int_n^{n+2}\omega'(t)dt-\int_{s+1}^{n+1}\omega'(t)dt\succ\\ &\left\{\int_{n+1}^{n+2}\omega'(t)dt-\int_n^{n+1}\omega'(t)dt\right\}+\left\{\int_n^{n+1}\omega'(t)dt-\int_{s+1}^{s+2}\omega'(t)dt\right\}\succ\\ &\qquad n^{-1+1/\delta}+\begin{cases}(n-s-1)n^{-1+1/\delta} & \text{if } s>n/2\\ n^{1/\delta} & \text{if } s\leq n/2.\end{cases}\end{aligned}$$

Relation (5.2) now follows. The case $s>n$ is treated in a similar way. $\square$

Let now $n>0$, $\lambda_n=r_ne^{i\theta_n}$. Then

$$\Bigl\|\frac{z^ne^{-in\theta_n}}{\|z^n\|_\varphi}-\mathbb{k}_{\lambda_n}\Bigr\|^2_\varphi$$
$$=\frac{\sum_{s\ge0,\,s\ne n}r_n^{2s}e^{-w_s}}{\sum_{s\ge0}r_n^{2s}e^{-w_s}}+\Bigl|e^{-w_n/2}-\frac{r_n^ne^{-w_n}}{(\sum_{s\ge0}r_n^{2s}e^{-w_s})^{1/2}}\Bigr|^2e^{w_n}$$
$$=S_1+S_2.$$

By (5.1),

$$S_1\le\sum_{s\ge0,\,s\ne n}\frac{c}{(n+1)^2(s+1)^2}\le\frac{c}{(n+1)^2}.$$

Furthermore,

$$S_2=\Bigl|1-\frac{r_n^ne^{-w_n/2}}{(\sum_{s\ge0}r_n^{2s}e^{-w_s})^{1/2}}\Bigr|^2\le1-\frac{r_n^{2n}e^{-w_n}}{\sum_{s\ge0}r_n^{2s}e^{-w_s}}=S_1.$$

Thus,

$$\sum_{n\ge0}\Bigl\|\frac{z^ne^{-in\theta_n}}{\|z^n\|_\varphi}-\mathbb{k}_{\lambda_n}\Bigr\|^2_\varphi<\infty.$$

By the Bari theorem (see [7, section A.5.7.1]), for some $N<\infty$, the linear operator U on $\mathcal{F}_\varphi$ determined by the equalities $U(z^n)=z^n$, $0\le n<N$, $U(z^n/\|z^n\|_\varphi)=\mathbb{k}_{\lambda_n}$, $n\ge N$, extends to an isomorphism. Therefore, the system

$$\{1,z,\dots,z^{N-1},\mathbb{k}_{\lambda_N},\mathbb{k}_{\lambda_{N+1}},\dots\}\tag{5.3}$$

is a Riesz basis in $\mathcal{F}_\varphi$.

Let $f\in\mathcal{F}_\varphi$ be orthogonal to all $\mathbb{k}_{\lambda_n}$, $n\ge0$. We define $g\in\mathcal{F}_\varphi$ by

$$g(z)=\prod_{1\le n<N}\frac{z}{z-\lambda_n}f(z).$$

Then g is orthogonal to all the elements of the Riesz basis (5.3) which is impossible. Therefore, the system $\{\mathbb{k}_{\lambda_n}\}_{n\ge0}$ is complete in $\mathcal{F}_\varphi$. Since the system $\{\mathbb{k}_{\lambda_n}\}_{n\ge N}$ is a Riesz basis in a subspace of codimension N, another application of the Bari theorem yields that the system $\{\mathbb{k}_{\lambda_n}\}_{n\ge0}$ is a Riesz basis in $\mathcal{F}_\varphi$. □

References

[1] A. Borichev, R. Dhuez, K. Kellay, *Sampling and interpolation in large Bergman and Fock spaces*, Journal of Functional Analysis **242** (2007), no. 2, 563–606.

[2] F. Holland, R. Rochberg, *Bergman kernel asymptotics for generalized Fock spaces*, J. Anal. Math. **83** (2001), 207–242.

[3] K. Isaev, R. Yulmukhametov, *The absence of unconditional bases of exponentials in Bergman spaces on non-polygonal domains*, Izvestiya RAN **71** (2007), 69–90.

[4] V. Lutsenko, *Unconditional bases of exponentials in Smirnov spaces*, Thesis, Ufa, 1992 (in Russian).

[5] Yu. Lyubarskii, E. Malinnikova, *On approximation of subharmonic functions*, J. Anal. Math. **83** (2001), 121–149.

[6] Yu. Lyubarskii, K. Seip, *Sampling and interpolation of entire functions and exponential systems in convex domains*, Ark. Mat. **32** (1994), 157–193.

[7] N. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz*, American Mathematical Society, Providence, RI, 2002.

[8] N. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.

[9] J. Ortega-Cerda, *Multipliers and weighted $\overline{\partial}$-estimates*, Rev. Mat. Iberoamericana **18** (2002), no. 2, 355–377.

[10] K. Seip, *Density theorems for sampling and interpolation in the Bargmann–Fock space*, I, J. Reine Angew. Math. **429** (1992) 91–106.

[11] R. Yulmukhametov, *Approximation of subharmonic functions.* (Russian) Anal. Math. **11** (1985), no. 3, 257–282.

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